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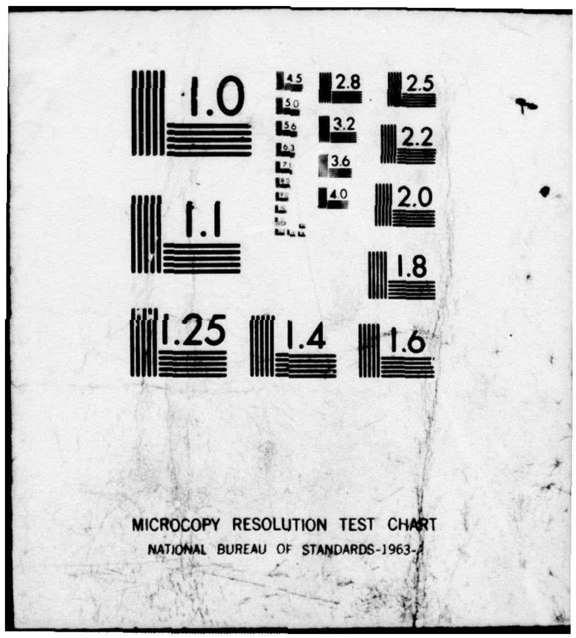
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COMBINATORIAL ASPECTS OF LINEAR ALGEBRA

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is a final report on Contract DAAG29-74-C-0007 covering research on combinatorial aspects of linear algebra.

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COMBINATORIAL ASPECTS OF LINEAR ALGEBRA

The basic theme of this contract has been the interplay between combinatorial theory and linear algebra, and most of the research has been on three aspects of that relation: polyhedral combinatorics, eigenvalues of matrices of combinatorial interest, and estimation (by combinatorial techniques) of eigenvalues of general matrices. There have been a few studies independent of these, most notably the paper on Worst-case analysis of linear DC circuits [14] (with Brayton and Scott) described in the report covering 9-30-75 to 6-30-76, but we will confine our discussion here to the three principal topics described above:

1. <u>Estimation of Eigenvalues</u>. The classic result on this problem is the theorem of Gersgorin: every eigenvalue of every complex matrix A is contained in the union of the disks: $|a_{kk}-\lambda| \leq \sum_{i\neq k} |a_{kj}|$. In [6], we characterized all sets of $n^2(n-1)$ nonnegative numbers $\{d_{ij}^k\}_{i\neq j}$ with the following property: $\forall$ matrix A of order n, every eigenvalue of A lies in the union of the n disks: $|a_{kk}-\lambda| \leq \sum_{i\neq j} d_{ij}^k |a_{ij}|$. Not only the original Gersgorin theorem, but all its known generalizations are corollaries of this characterization (which we do not describe here since its statement is somewhat complicated). The proof uses graph theory and Helly's theorem as well as other concepts from linear programming. The combinatorial methods used led to a proof of the following combinatorial theorem: a family of n convex closed pointed cones $C_1,\ldots,C_n$ in R^n satisfies $\cup C_i \cup -C_i = R^n$ if and only if $\forall \varepsilon_i = \pm 1$, $i=1,\ldots,n$, $\bigcap_i \varepsilon_i C_i \neq \{0\}$. We have proved the analogous theorem for n cones in C^n [9]. An interesting question for R^n was also provoked by this work. Let P be a polytope in R^n, let $t \geq 0$, and let $K_1,\ldots,K_r$ be closed convex sets in P such that every t-flat meeting P meets $\cup K_i$. Do there exist $P_i \subset K_i$, P_i a polytope, such that every t-flat meeting P meets $\cup P_i$? We proved this for t = 0, and J. Zaks (with others) has eventually found all triples (n,t,r) for which the statement is true.

2. <u>Polyhedral Combinatorics</u>. In this subject, the purpose is insight into combinatorial situations via linear programming. Generally the aim of the game is to prove that there are integer optima for appropriate primal and dual problems. In the early fifties, we participated in the initial stages of this program and clarified the useful role played by totally unimodular matrices in each work. If the linear programming matrix A is totally unimodular (every nonsingular square submatrix has determinant ± 1), and if right hand side and objective vector were both integral, then every vertex of primal and dual would be integral. Over the last fifteen years, owing principally to Jack Edmonds, the late Ray Fulkerson and their students, other instances of the integrality phenomenon (with resulting combinatorial consequences) were found where A is <u>not</u> totally unimodular. Beginning about three years ago, we tried to understand whether unimodularity still had a role to play. We found that, in many cases, it does. If $\bar{x}$ is a vertex of $\{x|Ax\leq b\}$, b is integral, and if the g.c.d. of the maximal minors of the submatrix of A found by those rows i such that $(A\bar{x})_i = b_i$ is 1, then the integrality of b and a theorem of H.J. Smith imply that $\bar{x}$ is integral. We named the phenomenon <u>local unimodularity</u>, distinguished three types of local unimodularity and showed how in many cases it existed for either primal or dual. It is still an open problem whether local unimodularity holds in the cases where we have not demonstrated it. We also showed how one could exploit the existence of optimal integral value for the dual objective (for all possible integral c) to demonstrate that the primal polyhedron had all vertices integral, a technique that has since become standard.

These insights enabled us to discover some new application of the concepts of polyhedral combinatorics. We generalized the max flow min cut theorem in several ways [3], gave generalizations of Greene and Kleitman's improvements of the Dilworth (and anti-Dilworth) theorem [17], and introduced the general concept of lattice polyhedron [18] [20]. Most of the theorems of polyhedral combinatorics of the last 15 years (with the notable exception of matching polytopes and perfect graphs) are contained in the lattice polyhedron approach. But the most interesting aspect of the approach is that there is a complete representation of lattice polyhedra where the right-hand side is 1, by looking at blockers and anti-blockers, and they are intimately related to paths in networks closed with respect to switching (blocker), and maximal chains in partially ordered sets (anti-blocker).

Finally, we mention our concept of superperfect graph (in our report for the period 1-1-74 to 3-31-74), which arose in the study of scheduling ship construction. Our viewpoint led to a heuristic algorithm (developed with Ellis Johnson) that has been useful in practice, and we have also used ideas of polyhedral combinatorics for a linear programming interpretation of superperfection.

3. <u>Eigenvalues of $\mathcal{G}$-Graphs</u>. Apart from the paper by Collatz and Sinogowitz appearing shortly after World War II, the subject of eigenvalues of graphs seems to have started with our work on the triangular association scheme and (with R. Singleton) on Moore graphs around 1960, both of which used eigenvalues as a tool of investigation for other problems. Most of the literature of several hundred articles and two monographs continues that spirit of using eigenvalues of the adjacency matrix to derive properties of the graph. Other than [12], [13], [19], most of our recent work has asked the question: what numerical functions of a graph have their magnitude related in at least a crude way to functions of the spectrum? By this, we mean: is it true that the graph function goes to infinity on a sequence of graphs if and only if the spectral function does [7]. We have found several instances of this phenomenon [8], [10], [16], and in the past year made two observations that clarified our vision on this class of problems. The first observation was that many of spectral function our results could be described in terms of the new concept of Ramsey function on a partially ordered set (see our report for 9-30-75 to 6-30-76). This concept seems to be relevant to many parts

of combinatorial mathematics, and we have already used it to clarify and extend results of Hell, Miller, Erdös and others.

Our second observation is that the methods we were using in the study of spectral functions could be tried (with the possibility, but not the guarantee of success) on matrices other than the (0,1) adjacency matrices of graphs. If $\mathcal{S}$ is any set of nonzero real numbers, let $A(\mathcal{S})$ be the set of all matrices whose nonzero entries are in $\mathcal{S}$; $\mathcal{S}$-graphs correspond to symmetric $\mathcal{S}$-matrices with 0 diagonal. Our best results to date are summarized in [21].

4. Other Activity. We mention that during the last three years, our students M. Gargano, P. Rolland and B. Jamil received their Ph.d.'s, the first two writing dissertations on eigenvalues of graphs. Rolland's thesis proved that, for all n, the tetrahedral association scheme is characterized by its parameters, contradicting a folk conjecture that there would be at least one exceptional value.

In addition, we note that we have given about thirty invited addresses on the work done under this contract during the last three years, including the 21st Conference of Army Mathematics at El Paso, and the International Congress of Mathematicians at Vancouver.

Reports Written and Papers Appearing in This Contract

[1] On eigenvalues of symmetric (+1,-1) matrices, RC 4571, Israel Journal of Mathematics 17(1974), 69-75.

[2] Self-orthogonal latin squares of all orders $n \neq 2,3,6$, Bulletin AMS 80(1974), 116-118.

[3] A generalization of max flow-min cut, RC 4607, Mathematical Programming 6 (1974), 352-359.

[4] Eigenvalues of graphs, RC 4688, appeared in Studies in Graph Theory, Carus Monograph of MAA, edited by D. R. Fulkerson, 1975.

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[6] Linear G-functions, RC 4701, Linear and Multilinear Algebra 3(1975), 45-52.

[7] Spectral functions of graphs, RC 4981, in Proceedings of the International Congress of Mathematicians 1974, Vol. 2, 461-464.

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[9] On convex cones in C^n, Bulletin of the Mathematical Institute of Taiwan 3(1975), 1-5.

[10] On graphs whose least eigenvalue exceeds $-1-\sqrt{2}$, RC 5184, to appear in Linear Algebra and its Applications.

[11] On spectrally bounded signed graphs, Transaction of the 21st Conference of Army Mathematicians, ARO Report 76-1, U.S. Army Research Office, Durham, 1-6.

[12] On the line graph of the complete tripartite graph (with B.A. Jamil), RC 5748, to appear in Linear and Multilinear Algebra.

[13] On the spectral radii of topologically equivalent graphs (with John Howard Smith), in Recent Advances in Graph Theory, edited by M. Fiedler, Czechoslovak Academy of Sciences (1975), 273-282.

[14] A theorem on inverses of convex sets of real matrices, with applications to the worst-case DC problem (with R.K. Brayton and T.R. Scott), RC 5811, to appear in IEEE Transactions on Circuit Theory.

[15] Local unimodularity in the matching polytope (with Rosa Oppenheim), RC 5813, to appear in Discrete Mathematics.

[16] On signed graphs and grammians, RC 5905, to appear in Geometrie Dedicata.

[17] On partitions of a partially ordered set (with D.S. Schwartz), RC 5961, to appear in Journal of Combinatorial Theory B.

[18] On lattice polyhedra (with D.S. Schwartz), RC 5979, to appear in Proceedings of Combinatorial Conference at Kezthely, Hungary.

[19] On limit points of the least eigenvalue of a graph, RC 5938, to appear in Ars Combinatoria.

[20] Lattice Polyhedra II: Construction and examples, RC 6268.

[21] Nearest $\mathcal{B}$-matrices of given rank and the Ramsey problem for bipartite $\mathcal{B}$-graphs (with Peter Joffe), RC 6061, to appear in Proceedings of Paris Conference on Combinatorics.

[22] Multicoloration dan les hypergraphes unimodulaires, et matrices dont les coefficients sont des ensembles (with C.R. Berge), RC 6294, to appear in Proceedings of Paris Conference on Combinatorics.

[23] Lattice Polyhedra III: Blockers and anti-blockers of lattice clutters, RC 6368, to appear in a Mathematical Programming Study.